

# RELATIVE OSCILLATION THEORY FOR STURM-LIOUVILLE OPERATORS EXTENDED

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**ABSTRACT.** We extend relative oscillation theory to the case of Sturm–Liouville operators  $Hu = r^{-1}(-(pu')' + qu)$  with different  $p$ 's. We show that the weighted number of zeros of Wronskians of certain solutions equals the value of Krein's spectral shift function inside essential spectral gaps.

## 1. INTRODUCTION

In [5] we have developed an analog of classical oscillation theory for Sturm–Liouville operators which, rather than measuring the spectrum of one single operator, measures the difference between the spectra of two different operators. Hence the name relative oscillation theory. The main idea behind this extension is to replace zeros of solutions of one operator by weighted zeros of Wronskians of solutions of two different operators. That zeros of the Wronskian are related to oscillation theory is indicated by an old paper of Leighton [6], who noted that if two solutions have a non-vanishing Wronskian, then their zeros must intertwine each other. Their use as an adequate tool for the investigation of the spectrum of one single operator is due to Gesztesy, Simon, and one of us [1].

The purpose of this paper is to extend relative oscillation theory for two different Sturm–Liouville equations

$$(1.1) \quad \tau_j = \frac{1}{r} \left( -\frac{d}{dx} p_j \frac{d}{dx} + q_j \right), \quad j = 0, 1.$$

In [5] we considered the case  $p_0 = p_1$ , here we want to extend relative oscillation theory to the case  $p_0 \neq p_1$ . In particular, for  $H_j$ ,  $j = 0, 1$ , self-adjoint operators associated with  $\tau_j$ , we want to show that the weighted number of zeros of Wronskians of certain solutions equals the value of Krein's spectral shift function  $\xi(\lambda, H_1, H_0)$  inside essential spectral gaps. To do this, and to make sure that the spectral shift function is well-defined, we will need to find a continuous path connecting the operators  $H_0$  and  $H_1$  in the metric introduced by the trace norm of resolvent differences.

In Section 2 we will recall the necessary background and fix our notation. Moreover, we will present the basic result for the case of regular operators. In Section 3 we have a quick look at Sturm's classical comparison theorem for zeros of solutions and its extension to zeros of Wronskians of solutions. Section 4 is concerned with relative oscillation theory for singular operators and contains our key result, Theorem 4.11, which connects the weighted zeros of Wronskians with Krein's spectral

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shift function. The remaining sections contain the proofs for our main results and our final appendix collects some facts on the spectral shift functions plus some abstract results which form the functional analytic core of the proof of our main theorem.

## 2. WEIGHTED ZEROS OF WRONSKIANS, PRÜFER ANGLES, AND REGULAR OPERATORS

We begin by fixing our notation and reviewing some simple facts from [5]. In particular, we refer to [5] for further details.

We will consider Sturm–Liouville operators on  $L^2((a, b), r dx)$  with  $-\infty \leq a < b \leq \infty$  of the form

$$(2.1) \quad \tau = \frac{1}{r} \left( -\frac{d}{dx} p \frac{d}{dx} + q \right),$$

where the coefficients  $p, q, r$  are real-valued satisfying

$$(2.2) \quad p^{-1}, q, r \in L^1_{loc}(a, b), \quad p, r > 0.$$

We will use  $\tau$  to describe the formal differentiation expression and  $H$  for the operator given by  $\tau$  with separated boundary conditions at  $a$  and/or  $b$ .

If  $a$  (resp.  $b$ ) is finite and  $q, p^{-1}, r$  are in addition integrable near  $a$  (resp.  $b$ ), we will say  $a$  (resp.  $b$ ) is a *regular* endpoint. We will say  $\tau$  respectively  $H$  is *regular* if both  $a$  and  $b$  are regular.

For every  $z \in \mathbb{C} \setminus \sigma_{ess}(H)$  there is a unique (up to a constant) solution  $\psi_-(z, x)$  of  $\tau u = zu$  which is in  $L^2$  near  $a$  and satisfies the boundary condition at  $a$  (if any). Similarly there is such a solution  $\psi_+(z, x)$  near  $b$ .

One of our main objects will be the (modified) Wronskian

$$(2.3) \quad W_x(u_0, u_1) = u_0(x) p_1(x) u_1'(x) - p_0(x) u_0'(x) u_1(x)$$

of two functions  $u_0, u_1$  and its zeros. Here we think of  $u_0$  and  $u_1$  as two solutions of two different Sturm–Liouville equations

$$(2.4) \quad \tau_j = \frac{1}{r} \left( -\frac{d}{dx} p_j \frac{d}{dx} + q_j \right), \quad j = 0, 1.$$

Under these assumptions  $W_x(u_0, u_1)$  is absolutely continuous and satisfies

$$(2.5) \quad W'(u_0, u_1) = (q_1 - q_0) u_0 u_1 + \left( \frac{1}{p_0} - \frac{1}{p_1} \right) p_0 u_0' p_1 u_1'.$$

Next we recall the definition of Prüfer variables  $\rho_u, \theta_u$  of an absolutely continuous function  $u$ :

$$(2.6) \quad u(x) = \rho_u(x) \sin(\theta_u(x)), \quad p(x) u'(x) = \rho_u(x) \cos(\theta_u(x)).$$

If  $(u(x), p(x) u'(x))$  is never  $(0, 0)$  and  $u, pu'$  are absolutely continuous, then  $\rho_u$  is positive and  $\theta_u$  is uniquely determined once a value of  $\theta_u(x_0)$  is chosen by requiring continuity of  $\theta_u$ .

Notice that

$$(2.7) \quad W_x(u, v) = -\rho_u(x) \rho_v(x) \sin(\Delta_{v,u}(x)), \quad \Delta_{v,u}(x) = \theta_v(x) - \theta_u(x).$$

Hence the Wronskian vanishes if and only if the two Prüfer angles differ by a multiple of  $\pi$ . We will call the total difference

$$(2.8) \quad \#_{(c,d)}(u_0, u_1) = \lceil \Delta_{1,0}(d)/\pi \rceil - \lfloor \Delta_{1,0}(c)/\pi \rfloor - 1$$

the number of weighted sign flips in  $(c, d)$ , where we have written  $\Delta_{1,0}(x) = \Delta_{u_1, u_0}$  for brevity.

We take two solutions  $u_j$ ,  $j = 1, 2$ , of  $\tau_j u_j = \lambda_j u_j$  and associated Prüfer variables  $\rho_j, \theta_j$ . Since we can replace  $q \rightarrow q - \lambda r$  it is no restriction to assume  $\lambda_0 = \lambda_1 = 0$ . We remark, that in (2.6) one has to take  $p_j$  as  $p$  for  $u_j$ ,  $j = 0, 1$ .

**Lemma 2.1.** *Abbreviate  $\Delta_{1,0}(x) = \theta_1(x) - \theta_0(x)$  and suppose  $\Delta_{1,0}(x_0) \equiv 0 \pmod{\pi}$ . If  $q_0(x) - q_1(x)$  and  $p_0(x) - p_1(x)$  are (i) negative, (ii) zero, or (iii) positive for a.e.  $x \in (x_0, x_0 + \varepsilon)$  respectively for a.e.  $x \in (x_0 - \varepsilon, x_0)$ , then the same holds true for  $(\Delta_{1,0}(x) - \Delta_{1,0}(x_0))/(x - x_0)$ .*

*Proof.* By (2.5) we have

$$\begin{aligned} W_x(u_0, u_1) &= -\rho_0(x)\rho_1(x)\sin(\Delta_{1,0}(x)) \\ (2.9) \quad &= -\int_{x_0}^x \left( (q_0(t) - q_1(t))u_0(t)u_1(t) + \left( \frac{1}{p_1(t)} - \frac{1}{p_0(t)} \right) p_0 u_0'(t) p_1 u_1'(t) \right) dt. \end{aligned}$$

Case (ii) follows. For (i) and (iii), first note that if  $u_j(x_0) = 0$ ,  $j = 0, 1$ , we have that  $u_j$  and  $p_j u_j'$ ,  $j = 0, 1$  have the same sign close to  $x_0$ , and thus the result follows.

Now, look at  $P(u_0, u_1) = \frac{u_0}{u_1} W(u_0, u_1)$  (compare (3.1) below) (resp.  $P(u_1, u_0)$ ) and note that  $u_0/u_1$  has constant sign near  $x_0$ . The result now follows using the fact that the derivate  $P'(u_0, u_1)$  is always negative by the Picone identity (3.2) below.  $\square$

Hence  $\#_{(c,d)}(u_0, u_1)$  counts the weighted sign flips of the Wronskian  $W_x(u_0, u_1)$ , where a sign flip is counted as  $+1$  if  $q_0 - q_1$  and  $p_0 - p_1$  are positive in a neighborhood of the sign flip, it is counted as  $-1$  if  $q_0 - q_1$  and  $p_0 - p_1$  are negative in a neighborhood of the sign flip. In particular, we obtain

**Lemma 2.2.** *Let  $u_0, u_1$  solve  $\tau_j u_j = 0$ ,  $j = 0, 1$ , where  $p_0 - p_1 \geq 0$  and  $q_0 - q_1 \geq 0$ . Then  $\#_{(a,b)}(u_0, u_1)$  equals the number sign flips of  $W(u_0, u_1)$  inside the interval  $(a, b)$ .*

Finally, we have the following extension of [5, Thm. 2.3] to the case  $p_0 \neq p_1$ .

**Theorem 2.3.** *Let  $H_0, H_1$  be regular Sturm–Liouville operators associated with (2.4) and the same boundary conditions at  $a$  and  $b$ . Then*

$$(2.10) \quad \dim \text{Ran } P_{(-\infty, \lambda_1)}(H_1) - \dim \text{Ran } P_{(-\infty, \lambda_0]}(H_0) = \#_{(a,b)}(\psi_{0,\pm}(\lambda_0), \psi_{1,\mp}(\lambda_1)).$$

The proof will be given in Section 5.

### 3. STURM'S COMPARISON THEOREM

One of the core ingredients of oscillation theory is Sturm's comparison theorem for zeros of solutions. We begin by recalling this classical result.

Let  $u_j$  solve  $\tau_j u_j = \lambda_j u_j$ , where without loss of generality we assume  $\lambda_0 = \lambda_1 = 0$ . For  $x$  with  $u_1(x) \neq 0$  we introduce

$$(3.1) \quad P_x(u_0, u_1) = \frac{u_0(x)}{u_1(x)} W_x(u_0, u_1) = -\rho_0^2(x) \frac{\sin(\theta_0(x)) \sin(\Delta_{1,0}(x))}{\sin(\theta_1(x))}.$$

Obviously  $P(u_0, u_1)$  is zero if either  $u_0$  or the Wronskian  $W(u_0, u_1)$  vanishes. Moreover, a straightforward computation, verifies the Picone identity (see [14, (2.6.4)])

$$(3.2) \quad P'(u_0, u_1) = (q_1 - q_0)u_0^2 + (p_1 - p_0)u_0'^2 - p_1 \left( u_0' - \frac{u_0 u_1'}{u_1} \right)^2,$$

which shows that  $P(u_0, u_1)$  is a nonincreasing function if  $q_1 \leq q_0$  and  $0 < p_1 \leq p_0$ .

**Theorem 3.1** (Sturm's Comparison theorem). *Let  $q_0 - q_1 \geq 0$ ,  $p_0 - p_1 \geq 0$ , with once strict inequality, and  $\tau_j u_j = 0$ ,  $j = 0, 1$ . Then between any two zeros of  $u_0$  or  $W(u_0, u_1)$ , there is a zero of  $u_1$ .*

*Similarly, between two zeros of  $u_1$ , which are not at the same time zeros of  $u_0$ , there is at least one zero of  $u_0$  or  $W(u_0, u_1)$ .*

*Proof.* Assume that  $u_1$  has no zero,  $P(u_0, u_1)$  would be well defined on the closed interval between the zeros, and be zero at its end points. This contradicts monotonicity of  $P(u_0, u_1)$ . The second claim is similar.  $\square$

Note that this version is slightly more general than the one usually found in the literature (cf., e.g., [14]) since it includes the case of zeros of Wronskians. For the case  $p_0 = p_1$  this was already pointed out in [1]. Moreover, in this case one can also allow zeros of the Wronskian at singular endpoints [1, Cor. 2.3].

Next, the comparison theorem for Wronskians from [5] carries over to the case  $p_0 \neq p_1$  without modifying the proof.

**Theorem 3.2** (Comparison theorem for Wronskians). *Suppose  $u_j$  satisfies  $\tau_j u_j = \lambda_j u_j$ ,  $j = 0, 1, 2$ , where  $\lambda_0 r - q_0 \leq \lambda_1 r - q_1 \leq \lambda_2 r - q_2$ ,  $p_0 \geq p_1 \geq p_2$ .*

*If  $c < d$  are two zeros of  $W_x(u_0, u_1)$  such that  $W_x(u_0, u_1)$  does not vanish identically, then there is at least one sign flip of  $W_x(u_0, u_2)$  in  $(c, d)$ . Similarly, if  $c < d$  are two zeros of  $W_x(u_1, u_2)$  such that  $W_x(u_1, u_2)$  does not vanish identically, then there is at least one sign flip of  $W_x(u_0, u_2)$  in  $(c, d)$ .*

#### 4. RELATIVE OSCILLATION THEORY

After these preparations we are now ready to extend relative oscillation theory to the case  $p_0 \neq p_1$ . Except for Lemma 4.7 and our key result Theorem 4.11, all results in this section are straightforward modifications of the analog results in [5] and hence we omit the corresponding proofs.

**Definition 4.1.** *For  $\tau_0, \tau_1$  possibly singular Sturm–Liouville operators as in (2.4) on  $(a, b)$ , we define*

$$(4.1) \quad \underline{\#}(u_0, u_1) = \liminf_{d \uparrow b, c \downarrow a} \#(c, d)(u_0, u_1) \quad \text{and} \quad \overline{\#}(u_0, u_1) = \limsup_{d \uparrow b, c \downarrow a} \#(c, d)(u_0, u_1),$$

where  $\tau_j u_j = \lambda_j u_j$ ,  $j = 0, 1$ .

*We say that  $\#(u_0, u_1)$  exists, if  $\overline{\#}(u_0, u_1) = \underline{\#}(u_0, u_1)$ , and write*

$$(4.2) \quad \#(u_0, u_1) = \overline{\#}(u_0, u_1) = \underline{\#}(u_0, u_1).$$

*in this case.*

By Lemma 2.1 one infers that  $\#(u_0, u_1)$  exists if  $p_0 - p_1$  and  $q_0 - \lambda_0 r - q_1 + \lambda_1 r$  have the same definite sign near the endpoints  $a$  and  $b$ .

**Theorem 4.2** (Triangle inequality for Wronskians). *Suppose  $u_j$ ,  $j = 0, 1, 2$  are given functions with  $u_j$ ,  $p_j u'_j$  absolutely continuous and  $(u_j(x), p_j(x)u'_j(x)) \neq (0, 0)$  for all  $x$ . Then*

$$(4.3) \quad \underline{\#}(u_0, u_1) + \underline{\#}(u_1, u_2) - 1 \leq \underline{\#}(u_0, u_2) \leq \underline{\#}(u_0, u_1) + \underline{\#}(u_1, u_2) + 1$$

and similarly for  $\underline{\#}$  replaced by  $\overline{\#}$ .

We recall that in classical oscillation theory  $\tau$  is called oscillatory if a solution of  $\tau u = 0$  has infinitely many zeros.

**Definition 4.3.** *We call  $\tau_1$  relatively nonoscillatory with respect to  $\tau_0$ , if the quantities  $\underline{\#}(u_0, u_1)$  and  $\overline{\#}(u_0, u_1)$  are finite for all solutions  $\tau_j u_j = 0$ ,  $j = 0, 1$ . We call  $\tau_1$  relatively oscillatory with respect to  $\tau_0$ , if one of the quantities  $\underline{\#}(u_0, u_1)$  or  $\overline{\#}(u_0, u_1)$  is infinite for some solutions  $\tau_j u_j = 0$ ,  $j = 0, 1$ .*

Note that this definition is in fact independent of the solutions chosen as a straightforward application of our triangle inequality (cf. Theorem 4.2) shows.

**Corollary 4.4.** *Let  $\tau_j u_j = \tau_j v_j = 0$ ,  $j = 0, 1$ . Then*

$$(4.4) \quad |\underline{\#}(u_0, u_1) - \underline{\#}(v_0, v_1)| \leq 4, \quad |\overline{\#}(u_0, u_1) - \overline{\#}(v_0, v_1)| \leq 4.$$

The bounds can be improved using our comparison theorem for Wronskians to be  $\leq 2$  in the case of perturbations of definite sign.

If  $\tau_0$  is nonoscillatory our definition reduces to the classical one.

**Lemma 4.5.** *Suppose  $\tau_0$  is a nonoscillatory operator, then  $\tau_1$  is relatively nonoscillatory (resp. oscillatory) with respect to  $\tau_0$ , if and only if  $\tau_1$  is nonoscillatory (resp. oscillatory).*

To demonstrate the usefulness of Definition 4.3, we now establish its connection with the spectra of self-adjoint operators associated with  $\tau_j$ ,  $j = 0, 1$ .

**Theorem 4.6.** *Let  $H_j$  be self-adjoint operators associated with  $\tau_j$ ,  $j = 0, 1$ . Then*

- (i)  $\tau_0 - \lambda_0$  is relatively nonoscillatory with respect to  $\tau_0 - \lambda_1$  if and only if  $\dim \text{Ran } P_{(\lambda_0, \lambda_1)}(H_0) < \infty$ .
- (ii) Suppose  $\dim \text{Ran } P_{(\lambda_0, \lambda_1)}(H_0) < \infty$  and  $\tau_1 - \lambda$  is relatively nonoscillatory with respect to  $\tau_0 - \lambda$  for one  $\lambda \in [\lambda_0, \lambda_1]$ . Then it is relatively nonoscillatory for all  $\lambda \in [\lambda_0, \lambda_1]$  if and only if  $\dim \text{Ran } P_{(\lambda_0, \lambda_1)}(H_1) < \infty$ .

For a practical application of this theorem one needs of course criteria when  $\tau_1 - \lambda$  is relatively nonoscillatory with respect to  $\tau_0 - \lambda$  for  $\lambda$  inside an essential spectral gap. Without loss of generality we only consider the case where one endpoint is regular.

**Lemma 4.7.** *Let  $H_0$  be bounded from below. Suppose  $a$  is regular ( $b$  singular) and*

- (i)  $\lim_{x \rightarrow b} r(x)^{-1}(q_0(x) - q_1(x)) = 0$ ,  $\frac{q_0}{r}$  is bounded near  $b$ , and
- (ii)  $\lim_{x \rightarrow b} p_1(x)p_0(x)^{-1} = 1$ .

Then  $\sigma_{\text{ess}}(H_0) = \sigma_{\text{ess}}(H_1)$  and  $\tau_1 - \lambda$  is relatively nonoscillatory with respect to  $\tau_0 - \lambda$  for every  $\lambda \in \mathbb{R} \setminus \sigma_{\text{ess}}(H_0)$ .

The analogous result holds for  $a$  singular and  $b$  regular.

The proof will be given in Section 5.

Our next task is to reveal the precise relation between the number of weighted sign flips and the spectra of  $H_1$  and  $H_0$ . The special case  $H_0 = H_1$  is covered by [1]:

**Theorem 4.8** ([1]). *Let  $H_0$  be a self-adjoint operator associated with  $\tau_0$  and suppose  $[\lambda_0, \lambda_1] \cap \sigma_{ess}(H_0) = \emptyset$ . Then*

$$(4.5) \quad \dim \operatorname{Ran} P_{(\lambda_0, \lambda_1)}(H_0) = \#(\psi_{0,\mp}(\lambda_0), \psi_{0,\pm}(\lambda_1)).$$

Combining this result with our triangle inequality already gives some rough estimates.

**Lemma 4.9.** *Let  $H_0, H_1$  be self-adjoint operators associated with  $\tau_0, \tau_1$ , respectively, and separated boundary conditions. Suppose that  $(\lambda_0, \lambda_1) \subseteq \mathbb{R} \setminus (\sigma_{ess}(H_0) \cup \sigma_{ess}(H_1))$ , then*

$$(4.6) \quad \begin{aligned} & \dim \operatorname{Ran} P_{(\lambda_0, \lambda_1)}(H_1) - \dim \operatorname{Ran} P_{(\lambda_0, \lambda_1)}(H_0) \\ & \leq \#(\psi_{1,\mp}(\lambda_1), \psi_{0,\pm}(\lambda_1)) - \overline{\#}(\psi_{1,\mp}(\lambda_0), \psi_{0,\pm}(\lambda_0)) + 2, \end{aligned}$$

respectively,

$$(4.7) \quad \begin{aligned} & \dim \operatorname{Ran} P_{(\lambda_0, \lambda_1)}(H_1) - \dim \operatorname{Ran} P_{(\lambda_0, \lambda_1)}(H_0) \\ & \geq \overline{\#}(\psi_{1,\mp}(\lambda_1), \psi_{0,\pm}(\lambda_1)) - \#(\psi_{1,\mp}(\lambda_0), \psi_{0,\pm}(\lambda_0)) - 2. \end{aligned}$$

To extend Theorem 2.3 to the singular case, we need to require the following hypothesis similar to [5, H.3.15].

**Hypothesis H.4.10.** *Suppose  $H_0$  and  $H_1$  are self-adjoint operators associated with  $\tau_0$  and  $\tau_1$  and separated boundary conditions (if any). Introduce*

$$(4.8) \quad \begin{aligned} & A_0 = \frac{1}{r}(rp_0)^{1/2} \frac{d}{dx}, \\ & \mathfrak{D}(A_0) = \{f \in L^2((a, b), r dx) | f \in AC_{loc}(a, b), \sqrt{p_0}f' \in L^2(a, b)\} \end{aligned}$$

- (i)  $r^{-1}q_0$  is infinitesimally form bounded with respect to  $A_0^*A_0$ .
- (ii)  $r^{-1}(q_1 - q_0)$  is infinitesimally form bounded with respect to  $H_0$ .
- (iii) There is a  $C_1 > 1$  such that  $C_1^{-1} \leq p_0(x)^{-1}p_1(x) \leq C_1$  for all  $x$ .
- (iv)  $r^{-1}|r(p_0 - p_1)|^{1/2} \frac{d}{dx} R_{H_0}(z)$  and  $|(r^{-1}(q_1 - q_0))^{1/2} R_{H_0}(z)|$  are Hilbert–Schmidt for one (and hence for all)  $z \in \rho(H_0)$ .

We note that the conditions of the last hypothesis are for example satisfied for periodic operators if the coefficients are continuous and  $p_0^{-1} - p_1^{-1}$  and  $q_0 - q_1$  are integrable.

It will be shown in Section 7 that these conditions ensure that we can interpolate between  $H_0$  and  $H_1$  using operators  $H_\varepsilon$ ,  $\varepsilon \in [0, 1]$ , such that the resolvent difference of  $H_0$  and  $H_\varepsilon$  is continuous in  $\varepsilon$  with respect to the trace norm. Hence we can fix the spectral shift function  $\xi(\lambda, H_1, H_0)$  by requiring  $\varepsilon \mapsto \xi(\lambda, H_\varepsilon, H_0)$  to be continuous in  $L^1(\mathbb{R}, (\lambda^2 + 1)^{-1} d\lambda)$ , where we of course set  $\xi(\lambda, H_0, H_0) = 0$  (see Lemma 7.7). While  $\xi$  is only defined a.e., it is constant on the intersection of the resolvent sets  $\mathbb{R} \cap \rho(H_0) \cap \rho(H_1)$ , and we will require it to be continuous there. In particular, note that by Weyl’s theorem the essential spectra of  $H_0$  and  $H_1$  are equal,  $\sigma_{ess}(H_0) = \sigma_{ess}(H_1)$ .

**Theorem 4.11.** *Let  $H_0, H_1$  satisfy Hypothesis 4.10. Then for every  $\lambda \in \rho(H_0) \cap \rho(H_1) \cap \mathbb{R}$ , we have*

$$(4.9) \quad \xi(\lambda, H_1, H_0) = \#(\psi_{0,\pm}(\lambda), \psi_{1,\mp}(\lambda)).$$

## 5. PROOFS OF LEMMA 4.7 AND THE REGULAR CASE

To prove Lemma 4.7, we need the following modification of [5, Lem. 3.9]:

**Lemma 5.1.** *Let  $(\lambda_0, \lambda_1) \subseteq \mathbb{R} \setminus \sigma_{\text{ess}}(H_0)$ ,  $\lambda \in (\lambda_0, \lambda_1)$ . If  $p_0 = p_1$  and  $\lambda_0 < r^{-1}(q_1 - q_0) - \lambda < \lambda_1$  (at least near singular endpoints), then  $\tau_1 - \lambda$  is relatively nonoscillatory with respect to  $\tau_0 - \lambda$ .*

*Proof.* Using our comparison theorem, we have that from  $\#(u_0(\lambda_0), u_0(\lambda_1)) < \infty$ , we obtain

$$\overline{\#}(u_0(\lambda_1), u_1(\lambda)) < \infty, \quad \underline{\#}(u_0(\lambda_0), u_1(\lambda)) > -\infty$$

now the result follows as in [5, Lem. 3.9] by

$$\overline{\#}(u_0(\lambda), u_1(\lambda)) \leq \#(u_0(\lambda), u_0(\lambda_1)) + \overline{\#}(u_0(\lambda_1), u_1(\lambda)) + 1$$

as follows from the triangle inequality for Wronskians ([5, Thm. 3.4]) and [5, Thm. 3.8 (i)].  $\square$

Our next proof will require the following resolvent relation for form perturbations. It is a special case from [3, Sect. VI.3] (see in particular equation (VI.3.10); compare also Sect. II.3. in [8]).

**Lemma 5.2.** *Let  $H_0$  be a self-adjoint operator which is bounded from below and let  $\lambda$  be below its spectrum. Let  $V$  be relatively form bounded with respect to  $H_0$  and with bound less than one. Then, we have that  $H = H_0 + V$  is self-adjoint and for its resolvent we have*

$$(5.1) \quad R_H(z) = R_{H_0}^{1/2}(\lambda)(1 - (z - \lambda)R_{H_0}(\lambda) + C)^{-1}R_{H_0}^{1/2}(\lambda).$$

Here  $C$  is the bounded operator associated with the quadratic form

$$(5.2) \quad \psi \mapsto \langle R_{H_0}^{1/2}(\lambda)\psi, V R_{H_0}^{1/2}(\lambda)\psi \rangle.$$

We remark, that here and in what follows sums of operators have to be understood as forms sums. Now we come to the

*Proof of Lemma 4.7.* We first show that  $\sigma_{\text{ess}}(H_0) = \sigma_{\text{ess}}(H_1)$ . First of all, note that imposing an additional Dirichlet boundary condition at some point  $b_n \in (a, b)$  implies that the resolvents of the original and the perturbed operator differ by a rank one perturbation (cf., e.g. [12]). Furthermore, the perturbed operator decomposes into a direct sum of two operators, one regular part on  $(a, b_n)$  and one singular part on  $(b_n, b)$ . Since the resolvent of a regular Sturm–Liouville operator is Hilbert–Schmidt, the only interesting part for the essential spectrum is the singular operator on  $(b_n, b)$ . Denote the corresponding operators by  $H_j^n$ ,  $j = 1, 2$ . (i.e.,  $H_j^n$  is  $H_j$  restricted to  $(b_n, b)$  with a Dirichlet boundary condition at  $b_n$ ). Then it suffices to show that the resolvent difference of  $H_1^n$  and  $H_0^n$  can be made arbitrarily small by choosing  $b_n$  close to  $b$ .

Recall the definition of  $A_0$  from (4.8) and note that since  $\frac{q_0}{r}$  is bounded (for  $b_n$  sufficiently large),  $A_0 R_{H_0}^{1/2}(-\lambda)$  is bounded for  $-\lambda < \sigma(H_0)$ . By virtue of Lemma 5.2 we conclude

$$R_{H_1^n}(\lambda) = R_{H_0^n}(\lambda)^{1/2}(1 + C^n)^{-1}R_{H_0^n}(\lambda)^{1/2}$$

for  $\lambda$  below the spectrum of  $H_0$ , where

$$C^n = (A_0^n R_{H_0^n}(\lambda)^{1/2})^* \frac{p_1 - p_0}{p_0} (A_0^n R_{H_0^n}(\lambda)^{1/2}) + R_{H_0^n}(\lambda)^{1/2} \frac{q_1 - q_0}{r} R_{H_0^n}(\lambda)^{1/2}$$

and  $A_0^n$  denotes the restriction of  $A_0$  to  $(b_n, b)$  with a Dirichlet boundary condition at  $b_n$ .

By assumption,  $\frac{p_1 - p_0}{p_0}$  respectively  $\frac{q_1 - q_0}{r}$  and thus  $\|C^n\|$  can be made arbitrarily small. Hence  $(1 + C^n)^{-1} \rightarrow 1$  and the first claim follows.

Now, we come to the proof of the relatively nonoscillation part. Our condition on  $p_1/p_0$  imply that

$$p_0(x)(1 - \varepsilon_-(y)) \leq p_1(x) \leq p_0(x)(1 + \varepsilon_+(y)), \quad x \geq y,$$

where

$$\varepsilon_{\pm}(y) = \pm \sup_{x \geq y} (\pm(p_1(x)/p_0(x) - 1)) \rightarrow 0, \quad y \rightarrow b.$$

Now it follows, from our comparison theorem, that solutions  $u_{\pm}$  of  $\tau_{\pm} u_{\pm} = 0$  on  $(y, b)$ , where

$$\tau_{\pm} = \frac{1}{r} \left( -\frac{d}{dx} (1 + \varepsilon_{\pm}(y)) p_0 \frac{d}{dx} + q_1 - \lambda r \right),$$

satisfy  $\#(u_-, u_0) \geq \#(u_1, u_0) \geq \#(u_+, u_0)$ . Since  $u_{\pm}$  also solve  $\tilde{\tau}_{\pm} u_{\pm} = 0$  on  $(y, b)$ , where

$$\tilde{\tau}_{\pm} = \frac{1}{r} \left( -\frac{d}{dx} p_0 \frac{d}{dx} + \frac{q_1 - \lambda r}{1 + \varepsilon_{\pm}(y)} \right),$$

the result follows from our previous lemma since

$$\frac{r^{-1} q_1 - \lambda}{1 + \varepsilon_{\pm}(y)} - (r^{-1} q_0 - \lambda) = \frac{r^{-1} (q_1 - q_0) - \varepsilon_{\pm}(y) (r^{-1} q_0 - \lambda)}{1 + \varepsilon_{\pm}(y)} \rightarrow 0$$

as  $y \rightarrow b$ .  $\square$

Our next aim is to prove Theorem 2.3. The main ingredient will be Prüfer variables and the formula (2.5) for the derivative of the Wronskian. Let us suppose that  $\tau_{0,1}$  are both regular at  $a$  and  $b$  with boundary conditions

$$(5.3) \quad \begin{aligned} \cos(\alpha) f(a) - \sin(\alpha) p_j(a) f'(a) &= 0 \\ \cos(\beta) f(b) - \sin(\beta) p_j(b) f'(b) &= 0 \end{aligned}, \quad j = 0, 1.$$

Abbreviate  $p_{\varepsilon} = p_0 + \varepsilon(p_1 - p_0)$ . Note that  $p_{\varepsilon}^{-1}$  is locally integrable, since  $p_{\varepsilon}^{-1} \leq \max(p_0^{-1}, p_1^{-1})$ . Hence we can choose  $\psi_{\pm, \varepsilon}(\lambda, x)$  such that  $\psi_{-, \varepsilon}(\lambda, a) = \sin(\alpha)$ ,  $p_{\varepsilon}(a) \psi'_{-, \varepsilon}(\lambda, a) = \cos(\alpha)$  respectively  $\psi_{+, \varepsilon}(\lambda, b) = \sin(\beta)$ ,  $p_{\varepsilon}(b) \psi'_{+, \varepsilon}(\lambda, b) = \cos(\beta)$ . In particular, we may choose

$$(5.4) \quad \theta_-(\lambda, a) = \alpha \in [0, \pi), \quad -\theta_+(\lambda, b) = \pi - \beta \in [0, \pi).$$

Next we introduce

$$(5.5) \quad \tau_{\varepsilon} = \tau_0 + \varepsilon(\tau_1 - \tau_0) = \frac{1}{r} \left( -\frac{d}{dx} p_{\varepsilon} \frac{d}{dx} + q_{\varepsilon} \right), \quad \begin{aligned} q_{\varepsilon} &= q_0 + \varepsilon(q_1 - q_0) \\ p_{\varepsilon} &= p_0 + \varepsilon(p_1 - p_0) \end{aligned}$$

and investigate the dependence with respect to  $\varepsilon \in [0, 1]$ .

If  $u_{\varepsilon}$  solves  $\tau_{\varepsilon} u_{\varepsilon} = 0$ , then the corresponding Prüfer angles satisfy

$$(5.6) \quad \dot{\theta}_{\varepsilon}(x) = -\frac{W_x(u_{\varepsilon}, \dot{u}_{\varepsilon})}{\rho_{\varepsilon}^2(x)},$$

where the dot denotes a derivative with respect to  $\varepsilon$ .



As in [5, Lem. 5.1], we obtain by integrating (2.5) and using this to evaluate the corresponding difference quotient the following lemma.

**Lemma 5.3.** *We have*

$$(5.7) \quad W_x(\psi_{\varepsilon,\pm}, \dot{\psi}_{\varepsilon,\pm}) = \begin{cases} \int_x^b (q_0(t) - q_1(t)) \psi_{\varepsilon,+}(t)^2 dt \\ \quad + \int_x^b (p_1^{-1}(t) - p_0^{-1}(t)) p_\varepsilon \psi'_{\varepsilon,+}(t)^2 dt \\ - \int_a^x (q_0(t) - q_1(t)) \psi_{\varepsilon,-}(t)^2 dt \\ \quad + \int_a^x (p_1^{-1}(t) - p_0^{-1}(t)) p_\varepsilon \psi'_{\varepsilon,-}(t)^2 dt, \end{cases}$$

where the dot denotes a derivative with respect to  $\varepsilon$ ,  $\psi_{\varepsilon,\pm}(x) = \psi_{\varepsilon,\pm}(0, x)$ , and  $p_\varepsilon = p_0 + \varepsilon(p_1 - p_0)$ .

Since we assumed  $a$  and  $b$  to be regular, all integrals exist.

Denote the Prüfer angles of  $\psi_{\varepsilon,\pm}(x) = \psi_{\varepsilon,\pm}(0, x)$  by  $\theta_{\varepsilon,\pm}(x)$ . The last lemma implies for  $q_0 - q_1 \geq 0$ ,  $p_0 - p_1 \geq 0$ , that

$$(5.8) \quad \dot{\theta}_{\varepsilon,+}(x) \leq 0, \quad \dot{\theta}_{\varepsilon,-}(x) \geq 0.$$

Now we are ready to investigate the associated operators  $H_0$  and  $H_1$ . In addition, we will choose the same boundary conditions for  $H_\varepsilon$  as for  $H_0$  and  $H_1$ . The next lemma follows as in [5, Lem. 5.2].

**Lemma 5.4.** *Suppose  $q_0 - q_1 \geq 0$ ,  $p_0 - p_1 \geq 0$  (resp. both  $\leq 0$ ). Then the eigenvalues of  $H_\varepsilon$  are analytic functions with respect to  $\varepsilon$  and they are decreasing (resp. increasing).*

In particular, this implies that  $\dim \text{Ran } P_{(-\infty, \lambda)}(H_\varepsilon)$  is continuous from below (resp. above) in  $\varepsilon$  for every  $\lambda$ . Now we are ready for the

*Proof of Theorem 2.3.* Without restriction it suffices to assume  $\lambda_0 = \lambda_1 = 0$  and to prove the result only for  $\#(\psi_{0,+}, \psi_{\varepsilon,-})$ .

We can split  $q_0 - q_1$ ,  $p_0 - p_1$  in the form

$$\begin{aligned} q_0 - q_1 &= q_+ - q_-, & q_+, q_- &\geq 0, \\ p_0 - p_1 &= p_+ - p_-, & p_+, p_- &\geq 0, \end{aligned}$$

and introduce the operator

$$\tau_- = \frac{1}{r} \left( -\frac{d}{dx}(p_0 - p_-) \frac{d}{dx} + (q_0 - q_-) \right).$$

Now  $\tau_-$  is a negative perturbation of  $\tau_0$  and  $\tau_1$  is a positive perturbation of  $\tau_-$ .

Furthermore, define  $\tau_\varepsilon$  by

$$\tau_\varepsilon = \begin{cases} \tau_0 + 2\varepsilon(\tau_- - \tau_0), & \varepsilon \in [0, 1/2] \\ \tau_- + 2(\varepsilon - 1/2)(\tau_1 - \tau_-), & \varepsilon \in [1/2, 1]. \end{cases}$$

Let us look at

$$N(\varepsilon) = \#(\psi_{0,+}, \psi_{\varepsilon,-}) = \lceil \Delta_\varepsilon(b)/\pi \rceil - \lfloor \Delta_\varepsilon(a)/\pi \rfloor - 1, \quad \Delta_\varepsilon(x) = \Delta_{\psi_{0,+}, \psi_{\varepsilon,-}}(x)$$

and consider  $\varepsilon \in [0, 1/2]$ . At the left boundary  $\Delta_\varepsilon(a)$  remains constant whereas at the right boundary  $\Delta_\varepsilon(b)$  is increasing by Lemma 5.3. Moreover, it hits a multiple of  $\pi$  whenever  $0 \in \sigma(H_\varepsilon)$ . So  $N(\varepsilon)$  is a piecewise constant function which is continuous

from below and jumps by one whenever  $0 \in \sigma(H_\varepsilon)$ . By Lemma 5.4 the same is true for

$$P(\varepsilon) = \dim \operatorname{Ran} P_{(-\infty, 0)}(H_\varepsilon) - \dim \operatorname{Ran} P_{(-\infty, 0]}(H_0)$$

and since we have  $N(0) = P(0)$ , we conclude  $N(\varepsilon) = P(\varepsilon)$  for all  $\varepsilon \in [0, 1/2]$ . To see the remaining case  $\varepsilon \in [1/2, 1]$ , simply replace increasing by decreasing and continuous from below by continuous from above.  $\square$

## 6. APPROXIMATION IN TRACE NORM

Now we begin with the result for singular operators by proving the case where  $q_1 - q_0$  and  $p_1 - p_0$  have compact support.

**Lemma 6.1.** *Let  $H_j$ ,  $j = 0, 1$ , be Sturm–Liouville operators on  $(a, b)$  associated with  $\tau_j$ , and suppose that  $r^{-1}(q_1 - q_0)$  and  $p_1 - p_0$  have support in a compact interval  $[c, d] \subseteq (a, b)$ , where  $a < c$  if  $a$  is singular and  $d < b$  if  $b$  is singular. Moreover, suppose  $H_0$  and  $H_1$  have the same boundary conditions (if any).*

*Suppose  $\lambda_0 < \inf \sigma_{\text{ess}}(H_0)$ . Then*

$$(6.1) \quad \dim \operatorname{Ran} P_{(-\infty, \lambda_0)}(H_1) - \dim \operatorname{Ran} P_{(-\infty, \lambda_0]}(H_0) = \#(\psi_{1,\mp}(\lambda_0), \psi_{0,\pm}(\lambda_0)).$$

*Suppose  $\sigma_{\text{ess}}(H_0) \cap [\lambda_0, \lambda_1] = \emptyset$ . Then*

$$(6.2) \quad \begin{aligned} & \dim \operatorname{Ran} P_{[\lambda_0, \lambda_1)}(H_1) - \dim \operatorname{Ran} P_{(\lambda_0, \lambda_1]}(H_0) \\ &= \#(\psi_{1,\mp}(\lambda_1), \psi_{0,\pm}(\lambda_1)) - \#(\psi_{1,\mp}(\lambda_0), \psi_{0,\pm}(\lambda_0)). \end{aligned}$$

*Proof.* Define  $H_\varepsilon = \varepsilon H_1 + (1 - \varepsilon)H_0$  as usual and observe that  $\psi_{\varepsilon,-}(z, x) = \psi_{0,-}(z, x)$  for  $x \leq c$  respectively  $\psi_{\varepsilon,+}(z, x) = \psi_{0,+}(z, x)$  for  $x \geq d$ . Furthermore,  $\psi_{\varepsilon,\pm}(z, x)$  is analytic with respect to  $\varepsilon$  and  $\lambda \in \sigma_p(H_\varepsilon)$  if and only if  $W_d(\psi_{0,+}(\lambda), \psi_{\varepsilon,-}(\lambda)) = 0$ . Now the proof can be done as in the regular case.  $\square$

**Lemma 6.2.** *Suppose  $H_0, H_1$  satisfy the same assumptions as in the previous lemma and that there is a constant  $C_1 > 1$  such that  $C_1^{-1} \leq p_1(x)p_0(x)^{-1} \leq C_1$  for all  $x \in (a, b)$ . Furthermore, set  $H_\varepsilon = \varepsilon H_1 + (1 - \varepsilon)H_0$ . Then*

$$(6.3) \quad \|\sqrt{r^{-1}|q_0 - q_1|} R_{H_\varepsilon}(z)\|_{\mathcal{J}_2} \leq C(z), \quad \varepsilon \in [0, 1],$$

and

$$(6.4) \quad \|\sqrt{|p_1 - p_0|} \frac{d}{dx} R_{H_\varepsilon}(z)\|_{\mathcal{J}_2} \leq C(z), \quad \varepsilon \in [0, 1].$$

*In particular,  $H_0$  and  $H_1$  are resolvent comparable and*

$$(6.5) \quad \xi(\lambda, H_1, H_0) = \#(\psi_{1,\mp}(\lambda), \psi_{0,\pm}(\lambda))$$

*for every  $\lambda \in \mathbb{R} \setminus (\sigma(H_0) \cup \sigma(H_1))$ . Here  $\xi(H_1, H_0)$  is assumed to be constructed such that  $\varepsilon \mapsto \xi(H_\varepsilon, H_0)$  is a continuous mapping  $[0, 1] \rightarrow L^1((\lambda^2 + 1)^{-1} d\lambda)$ .*

*Proof.* Denote by

$$G_\varepsilon(z, x, y) = (H_\varepsilon - z)^{-1}(x, y) = \frac{\psi_{\varepsilon,-}(z, x_{<}) \psi_{\varepsilon,+}(z, y_{>})}{W(\psi_{\varepsilon,-}(z), \psi_{\varepsilon,+}(z))},$$

where  $x_{<} = \min(x, y)$ ,  $y_{>} = \max(x, y)$ , the Green's function of  $H_\varepsilon$ . As pointed out in the proof of the previous lemma,  $\psi_{\varepsilon,\pm}(z, x)$  is analytic with respect to  $\varepsilon$  and hence a simple estimate shows

$$\int_a^b \int_a^b |G_\varepsilon(z, x, y)|^2 |r(y)^{-1}(q_1(y) - q_0(y))| r(x) dx r(y) dy \leq C(z)^2$$

for  $\varepsilon \in [0, 1]$ , which establishes the first claim.

For the second claim, we need to show that

$$\begin{aligned} & \int_a^b \int_a^b |p_\varepsilon \partial_x G_\varepsilon(z, x, y)|^2 \left| \frac{p_1(x) - p_0(x)}{p_\varepsilon^2(x)} \right| r(x) dx r(y) dy \\ & \leq C(z) \int_c^d \left| \frac{p_1(x)p_0(x)}{p_\varepsilon^2(x)} \right| |p_0^{-1}(x) - p_1^{-1}(x)| r(x) dx \end{aligned}$$

is uniformly bounded in  $\varepsilon \in [0, 1]$ . However, this follows here from the integrand being integrable, since

$$0 < \frac{p_0}{p_\varepsilon} \leq C_1, \quad 0 < \frac{p_1}{p_\varepsilon} \leq C_1.$$

Moreover, a straightforward calculation (using (2.5)) and

$$\psi_{+, \varepsilon}(c) = \psi_{+, \varepsilon'}(c) W_c(\psi_{+, \varepsilon}, \psi_{-, \varepsilon'}) - \psi_{-, \varepsilon'}(c) W_c(\psi_{+, \varepsilon}, \psi_{+, \varepsilon'})$$

shows

$$\begin{aligned} G_{\varepsilon'}(z, x, y) &= G_\varepsilon(z, x, y) \\ &+ (\varepsilon - \varepsilon') \int_a^b G_{\varepsilon'}(z, x, t) r^{-1}(t) (q_1(t) - q_0(t)) G_\varepsilon(z, t, y) r(t) dt \\ &+ (\varepsilon - \varepsilon') \int_a^b \frac{\partial G_{\varepsilon'}(z, x, t)}{\partial t} r^{-1}(t) (p_1(t) - p_0(t)) \frac{\partial G_\varepsilon(z, t, y)}{\partial t} r(t) dt. \end{aligned}$$

Hence  $R_{H_{\varepsilon'}}(z) - R_{H_\varepsilon}(z)$  can be written as the sum of two products of two Hilbert-Schmidt operators, whose norm can be estimated by the first claims:

$$(6.6) \quad \|R_{H_{\varepsilon'}}(z) - R_{H_\varepsilon}(z)\|_{\mathcal{J}_1} \leq |\varepsilon' - \varepsilon| C(z)^2.$$

Thus  $\varepsilon \mapsto \xi(H_\varepsilon, H_0)$  is continuous. The rest follows from (7.4).  $\square$

Before proving Theorem 4.11, we still need to transform Hypothesis 4.10 in a form such that we can apply our operator theoretic results from the appendix. The next lemma will do the job.

**Lemma 6.3.** *Assume Hypothesis 4.10, and introduce*

$$(6.7) \quad \mathfrak{Q} = \{f \in L^2((a, b), r dx) \mid f \in AC_{loc}(a, b), \sqrt{p_0} f' \in L^2(a, b)\}.$$

Furthermore, introduce the following operators on  $\mathfrak{Q}$  with  $N = \lceil \sup_x (p_1(x)p_0(x)^{-1} - 1)_+ \rceil + 1$

$$(6.8) \quad A_j = \frac{1}{N^{1/2}} (p_0 - p_1)_+^{1/2} \frac{d}{dx}, \quad j = 1, \dots, N,$$

$$(6.9) \quad A_{N+1} = |q_0 - q_1|^{1/2}, \quad A_{N+2} = (p_0 - p_1)_-^{1/2} \frac{d}{dx}$$

$$(6.10) \quad S_1, \dots, S_N = 1, \quad S_{N+1} = \operatorname{sgn}(q_0 - q_1), \quad S_{N+2} = -1$$

Then Hypothesis 7.4 is satisfied with these operators and  $H_0, H_1$  are self-adjoint extensions of  $\tau_0, \tau_1$ , respectively.

*Proof.* By Lemma 7.6, it is sufficient to check the form bounds with respect to the form of  $\tau_0$  with  $q_0 = 0$ , since we have by [5, Lem. 4.1], that  $q_0, q_1$  will be infinitesimally form bounded.

To see the claims on the other operators, note that  $p_0^{1/2} f' \in L^2$  implies  $|p_1 - p_0|^{1/2} f' \in L^2$ , since  $p_0^{-1/2} |p_1 - p_0|^{1/2}$  is essentially bounded by assumption. We are left with computing the form bounds, but again ( $1 \leq j \leq N, u \in \mathfrak{Q}$ )

$$\|A_j u\|^2 = \frac{1}{N} \|p_0^{-1/2} (p_0 - p_1)_+^{1/2} p_0^{1/2} u'\|^2 \leq \frac{\sup_x (p_0(x)^{-1} p_1(x) - 1)_+}{N} \langle u, A_0^* A_0 u \rangle$$

which shows that the form bound with respect to  $A_0^* A_0$  is less than one. By Lemma 7.6 the same is true with respect to  $H_0$ .

Boundedness from below follows by noting, that the quadratic forms are bounded from below, by the bounds on  $q_0$  (resp.  $q_1$ ).  $\square$

Now we come to the

*Proof of Theorem 4.11.* We first assume that we have compact support near one endpoint, say  $a$ . Define by  $K_\varepsilon$  the multiplication operator by  $\chi_{(a, b_\varepsilon]}$  with  $b_\varepsilon \uparrow b$ . Then  $K_\varepsilon$  satisfies the assumptions of Lemma 7.7. The last lemma guarantees that Hypothesis 4.10 implies Hypothesis 7.4, so we can apply Lemma 7.3 by Lemma 7.7.

Denote by

$$\tau_\varepsilon = \frac{1}{r} \left( -\frac{d}{dx} p_\varepsilon \frac{d}{dx} + q_\varepsilon \right), \quad \begin{aligned} p_\varepsilon &= p_0 + \chi_{(a, b_\varepsilon]} (p_1 - p_0) \\ q_\varepsilon &= q_0 + \chi_{(a, b_\varepsilon]} (q_1 - q_0) \end{aligned}$$

and by  $\psi_{\varepsilon,-}$  the corresponding solutions satisfying the boundary condition at  $a$ . By Lemma 7.7 we have that  $\xi(H_\varepsilon, H_0)$  is constant and equal to  $\xi(H_1, H_0)$  once  $\varepsilon$  is greater than some  $\varepsilon_0$ .

Now let us turn to the Wronskians. We first prove the  $\#(\psi_{1,-}(\lambda), \psi_{0,+}(\lambda))$  case. By Lemma 6.2 we know

$$\xi(\lambda, H_\varepsilon, H_0) = \#(\psi_{\varepsilon,-}(\lambda), \psi_{0,+}(\lambda))$$

for every  $\varepsilon < 1$ . Concerning the right-hand side observe that

$$W_x(\psi_{\varepsilon,-}(\lambda), \psi_{0,+}(\lambda)) = W_x(\psi_{1,-}(\lambda), \psi_{0,+}(\lambda))$$

for  $x \leq b_\varepsilon$  and that  $W_x(\psi_{\varepsilon,-}(\lambda), \psi_{0,+}(\lambda))$  is constant for  $x \geq b_\varepsilon$ . This implies that for  $\varepsilon \geq \varepsilon_0$  we have

$$\begin{aligned} \xi(\lambda, H_1, H_0) &= \xi(\lambda, H_\varepsilon, H_0) = \#(\psi_{\varepsilon,-}(\lambda), \psi_{0,+}(\lambda)) \\ &= \#_{(a, b_\varepsilon)}(\psi_{\varepsilon,-}(\lambda), \psi_{0,+}(\lambda)) = \#_{(a, b_\varepsilon)}(\psi_{1,-}(\lambda), \psi_{0,+}(\lambda)). \end{aligned}$$

In particular, the last item  $\#_{(a, b_\varepsilon)}(\psi_{1,-}(\lambda), \psi_{0,+}(\lambda))$  is eventually constant and thus has a limit which, by definition, is  $\#(\psi_{1,-}(\lambda), \psi_{0,+}(\lambda))$ .

For the corresponding  $\#(\psi_{1,+}(\lambda), \psi_{0,-}(\lambda))$  case one simply exchanges the roles of  $H_0$  and  $H_1$ .

Hence the result holds if the perturbation has compact support near one endpoint. Now one repeats the argument to remove the compact support assumption near the other endpoint as well.  $\square$

## 7. APPENDIX: THE SPECTRAL SHIFT FUNCTION

In this appendix we collect some facts on Krein's spectral shift function which are of relevance to us. Most results are taken from [13] (see also [10] for an easy introduction). The first part closely follows the appendix of [5].

Two operators  $H_0$  and  $H_1$  are called resolvent comparable, if

$$(7.1) \quad R_{H_1}(z) - R_{H_0}(z)$$

is trace class for one  $z \in \rho(H_1) \cap \rho(H_0)$ . By the first resolvent identity (7.1) then holds for all  $z \in \rho(H_1) \cap \rho(H_0)$ .

**Theorem 7.1** (Krein [4]). *Let  $H_1$  and  $H_0$  be two resolvent comparable self-adjoint operators, then there exists a function*

$$(7.2) \quad \xi(\lambda, H_1, H_0) \in L^1(\mathbb{R}, (\lambda^2 + 1)^{-1} d\lambda)$$

such that

$$(7.3) \quad \text{tr}(f(H_1) - f(H_0)) = \int_{-\infty}^{\infty} \xi(\lambda, H_1, H_0) f'(\lambda) d\lambda$$

for every smooth function  $f$  with compact support.

Note: Equation (7.3) holds in fact for a much larger class of functions  $f$ . See [13, Thm. 9.7.1] for this and a proof of the last theorem.

The function  $\xi(\lambda) = \xi(\lambda, H_1, H_0)$  is called Krein's spectral shift function and is unique up to a constant. Moreover,  $\xi(\lambda)$  is clearly constant on every interval  $(\lambda_0, \lambda_1) \subset \rho(H_0) \cap \rho(H_1)$ . Hence, if  $\dim \text{Ran } P_{(\lambda_0, \lambda_1)}(H_j) < \infty$ ,  $j = 0, 1$ , then  $\xi(\lambda)$  is a step function and

$$(7.4) \quad \dim \text{Ran } P_{(\lambda_0, \lambda_1)}(H_1) - \dim \text{Ran } P_{(\lambda_0, \lambda_1)}(H_0) = \lim_{\varepsilon \downarrow 0} (\xi(\lambda_1 - \varepsilon) - \xi(\lambda_0 + \varepsilon)).$$

This formula clearly explains the name spectral shift function.

Before investigating further the properties of the SSF, we will recall a few things about trace ideals (see for example [9]). First, for  $1 \leq p < \infty$  denote by  $\mathcal{J}^p$  the Schatten  $p$ -class, and by  $\|\cdot\|_{\mathcal{J}^p}$  its norm. We will use  $\|\cdot\|$  for the usual operator norm. Using  $\|A\|_{\mathcal{J}^p} = \infty$  if  $A \notin \mathcal{J}^p$ , we have the following inequalities for all operators:

$$\|AB\|_{\mathcal{J}^p} \leq \|A\| \|B\|_{\mathcal{J}^p}, \quad \|AB\|_{\mathcal{J}^1} \leq \|A\|_{\mathcal{J}^2} \|B\|_{\mathcal{J}^2}.$$

Furthermore, we will use the notation of  $\mathcal{J}^p$ -converges to denote convergence in the respective norm. The following result from [2, Thm. IV.11.3] will be needed.

**Lemma 7.2.** *Let  $p > 0$ ,  $A_n \xrightarrow{\mathcal{J}^p} A$ ,  $T_n \xrightarrow{s} T$ ,  $S_n \xrightarrow{s} S$  sequences of strongly convergent bounded linear operators, then:*

$$(7.5) \quad \|T_n A_n S_n^* - T A S^*\|_{\mathcal{J}^p} \rightarrow 0.$$

Here  $\|\cdot\|_{\mathcal{J}^p}$  are the norms of the Schatten  $p$ -classes  $\mathcal{J}^p$ .

We will also need the following continuity result for  $\xi$ . It will also allow us to fix the unknown constant. The second part is [5, Lem. 7.3], the first from [13].

**Lemma 7.3.** *Suppose  $H_\varepsilon$ ,  $\varepsilon \in [0, 1]$ , is a family of self-adjoint operators, which is continuous in the metric*

$$(7.6) \quad \rho(A, B) = \|R_A(z_0) - R_B(z_0)\|_{\mathcal{J}^1}$$

for some fixed  $z_0 \in \mathbb{C} \setminus \mathbb{R}$  and abbreviate  $\xi_\varepsilon = \xi(H_\varepsilon, H_0)$ . Then there exists a unique choice of  $\xi_\varepsilon$  such that  $\varepsilon \mapsto \xi_\varepsilon$  is continuous  $[0, 1] \rightarrow L^1(\mathbb{R}, (\lambda^2 + 1)^{-1} d\lambda)$  with  $\xi_0 = 0$ .

If  $H_\varepsilon \geq \lambda_0$  is bounded from below, we can also allow  $z_0 \in (-\infty, \lambda_0)$ .

For  $\lambda \in \rho(H_1) \cap \mathbb{R}$ , we have that there is an  $\varepsilon_0$  such that  $\xi_\varepsilon(\lambda) = \xi_1(\lambda)$  for  $\varepsilon > \varepsilon_0$ .

*Proof.* We just need to proof the third part. For  $\varepsilon$  close to 1 a whole neighborhood of  $\lambda$  is in  $\rho(H_\varepsilon) \cap \mathbb{R}$ , since the resolvent sets converge. Furthermore, we know from this that the  $\xi_\varepsilon$  is integer valued near 1 in a neighborhood of  $\lambda$ . Now the claim follows from the convergence of  $\xi_\varepsilon \rightarrow \xi_1$  in  $L^1(\mathbb{R}, (\lambda^2 + 1)^{-1} d\lambda)$ .  $\square$

Our final aim is to find some conditions which allow us to verify the assumptions of this lemma. To do this, we derive some properties of relatively bounded operators multiplied by strongly continuous families of operators.

**Hypothesis H.7.4.** Suppose  $H_0$  is self-adjoint and bounded from below. Let  $A_j$ ,  $j = 1, \dots, n$ , be closed operators and  $S_j$ ,  $j = 1, \dots, n$  be bounded operators with  $\|S_j\| \leq 1$ . Furthermore, suppose that these satisfy for  $j = 1, \dots, n-1$  that

- (i)  $A_j^* A_j$  is relatively form bounded with respect to  $H_0$  with relative form bound less than one and  $S_j$  is positive, or
- (i')  $A_j^* A_j$  is infinitesimally form bounded with respect to  $H_0$ .

Suppose for  $j = n$ , that

- (ii)  $A_j^* A_j$  is relatively form bounded with respect to  $H_0$  with relative form bound less than one.

Note that condition (i) implies that  $A_j^* S_j A_j$  is a positive operator.

We recall that  $A^* A$  being form bounded with respect to  $H_0$  means that we have  $\mathfrak{Q}(A^* A) \supseteq \mathfrak{Q}(H_0)$  and

$$(7.7) \quad \langle \psi, A^* A \psi \rangle \leq a \langle \psi, H_0 \psi \rangle + b \|\psi\|^2, \quad \forall \psi \in \mathfrak{Q}(H_0).$$

for some  $0 \leq a < 1$ ,  $0 \leq b$ . The form bound is the infimum over all  $a$  such that (7.7) holds.

The next lemma is modified from [5, Lem. 7.5], to be able to deal with differential operators and sums of operators.

**Lemma 7.5.** Let  $\varepsilon \ni [0, 1] \rightarrow K_\varepsilon$  be a strongly continuous family of self-adjoint bounded operators which satisfy  $0 = K_0 \leq K_\varepsilon \leq K_1 = 1$ .

Let

- (i)  $\varepsilon \mapsto H_\varepsilon$  satisfy the assumptions of Lemma 7.3,
- (ii)  $S$  be a bounded operator with  $\|S\| \leq 1$ , and
- (iii)  $A$  be a closed operator such that  $A^* A$  is relatively bounded with respect to  $H_\varepsilon$  with uniform in  $\varepsilon$  bound less than one, and  $AR_{H_\varepsilon}(z) \in \mathcal{J}^2$  for one  $z \in \mathbb{C} \setminus \mathbb{R}$ .

Then  $\tilde{H}_\varepsilon = H_\varepsilon + A^* K_\varepsilon S A$  also satisfies the assumptions of Lemma 7.3. Furthermore, for form bounded  $B$  with  $BR_{H_\varepsilon}(z) \in \mathcal{J}^2$ , we have  $BR_{\tilde{H}_\varepsilon}(z) \in \mathcal{J}^2$  for all  $\varepsilon \in [0, 1]$ .

*Proof.* We will abbreviate  $V_\varepsilon = A^* K_\varepsilon S A$ ,  $\tilde{H}_\varepsilon = H_\varepsilon + V_\varepsilon$ ,  $R_\varepsilon(z) = R_{H_\varepsilon}(z)$ , and  $\tilde{R}_\varepsilon(z) = R_{\tilde{H}_\varepsilon}(z)$ . By the KLMN theorem ([7, Thm. X.17]),  $\tilde{H}_\varepsilon$  is self-adjoint since

$$|\langle \psi, V_\varepsilon \psi \rangle| \leq |\langle A \psi, K_\varepsilon S A \psi \rangle| \leq \langle \psi, A^* A \psi \rangle, \quad \psi \in \mathfrak{Q}(V_\varepsilon).$$

Moreover, using (7.7) we obtain

$$\|AR_\varepsilon(-\lambda)^{1/2}\|^2 \leq a, \quad \frac{b}{a} < \lambda.$$

For  $\lambda > \frac{b}{a}$  we have by Lemma 5.2

$$\begin{aligned} \tilde{R}_\varepsilon(-\lambda) &= R_\varepsilon(-\lambda)^{1/2}(1 + C_\varepsilon)^{-1}R_\varepsilon(-\lambda)^{1/2}, \\ C_\varepsilon &= (AR_\varepsilon(-\lambda)^{1/2})^*(K_\varepsilon SAR_\varepsilon(-\lambda)^{1/2}). \end{aligned}$$

Hence, a straightforward calculation shows

$$\begin{aligned} (7.8) \quad \tilde{R}_\varepsilon(-\lambda) &= R_\varepsilon(-\lambda) - (AR_\varepsilon(-\lambda))^*(1 + \tilde{C}_\varepsilon)^{-1}(K_\varepsilon SAR_\varepsilon(-\lambda)), \\ \tilde{C}_\varepsilon &= (K_\varepsilon SAR_\varepsilon(-\lambda)^{1/2})(AR_\varepsilon(-\lambda)^{1/2})^*. \end{aligned}$$

By  $\|\tilde{C}_\varepsilon\| \leq a < 1$ , we have that  $(1 + \tilde{C}_\varepsilon)^{-1}$  exists. Furthermore, note that (7.8) implies, that  $B\tilde{R}_\varepsilon(-\lambda) \in \mathcal{J}^2$ , since:

$$B\tilde{R}_\varepsilon(-\lambda) = BR_\varepsilon(-\lambda) - BR_\varepsilon(-\lambda)^{1/2}(AR_\varepsilon(-\lambda)^{1/2})^*(1 + \tilde{C}_\varepsilon)^{-1}(K_\varepsilon SAR_\varepsilon(-\lambda)),$$

and  $AR_\varepsilon(-\lambda) \in \mathcal{J}^2$ . Now, look at

$$\begin{aligned} D_{\varepsilon, \varepsilon'}\psi &= (-\tilde{C}_\varepsilon(1 + \tilde{C}_\varepsilon)^{-1} - \tilde{C}_{\varepsilon'}(1 + \tilde{C}_{\varepsilon'})^{-1})\psi \\ &= (C_{\varepsilon'} - C_\varepsilon)(1 + C_\varepsilon)^{-1}\psi - C_{\varepsilon'}D_{\varepsilon, \varepsilon'}\psi, \end{aligned}$$

where

$$D_{\varepsilon, \varepsilon'} = (1 + \tilde{C}_\varepsilon)^{-1} - (1 + \tilde{C}_{\varepsilon'})^{-1}.$$

Taking norms we obtain

$$\|D_{\varepsilon, \varepsilon'}\psi\| = \frac{1}{1-a} \|(C_{\varepsilon'} - C_\varepsilon)(1 + C_\varepsilon)^{-1}\psi\|,$$

where the last term converges to 0 as  $\varepsilon' \rightarrow \varepsilon$ . This implies, that  $(1 + \tilde{C}_\varepsilon)^{-1}$  is strongly continuous. Now, we obtain from (7.8) for the difference of resolvents

$$\tilde{R}_\varepsilon(-\lambda) - \tilde{R}_{\varepsilon'}(-\lambda) = (AR_\varepsilon(-\lambda))^*((1 + \tilde{C}_\varepsilon)^{-1}K_\varepsilon - (1 + \tilde{C}_{\varepsilon'})^{-1}K_{\varepsilon'})(SAR_\varepsilon(-\lambda))$$

$\mathcal{J}^1$ -converges to 0 as  $\varepsilon \rightarrow \varepsilon'$  by Lemma 7.2 and by  $AR_\varepsilon(-\lambda) \in \mathcal{J}^2$ . This way we also obtain that  $\tilde{H}_\varepsilon$  and  $\tilde{H}_{\varepsilon'}$  are indeed resolvent comparable.  $\square$

We also recall the following well-known fact on quadratic forms:

**Lemma 7.6.** *Let  $v, s, t$  be quadratic forms, such that  $s$  is positive and symmetric, and  $v$  is infinitesimal form bounded with respect to  $s$ , and  $t$  is form bounded with bound less than 1 with respect to  $s$ . Then  $t$  is also form bounded with bound less than 1 with respect to  $s + v$ .*

*Proof.* Using  $|v(\psi)| \leq \varepsilon s(\psi) + C\|\psi\|^2$  for arbitrary small  $\varepsilon > 0$ , a direct calculation shows

$$s(\psi) \leq \frac{1}{1-\varepsilon}|s(\psi) + v(\psi)| + \frac{C(\varepsilon)}{1-\varepsilon}\|\psi\|^2.$$

Denoting by  $a$  the  $s$  bound of  $t$ , it follows that  $t$  is  $s + v$  bounded with bound less than  $a/(1-\varepsilon)$ , implying that the bound is again less than one.  $\square$

**Lemma 7.7.** *Let  $\varepsilon \ni [0, 1] \rightarrow K_\varepsilon$  be a strongly continuous family of self-adjoint bounded operators which satisfy  $0 = K_0 \leq K_\varepsilon \leq K_1 = 1$ .*

*Assume Hypothesis 7.4. Then*

$$(7.9) \quad H_\varepsilon = H_0 + \sum_{j=1}^n A_j^* K_\varepsilon S_j A_j$$

*are self-adjoint operators such that the assumptions of Lemma 7.3 hold.*

*Proof.* Introduce  $H_\varepsilon^m = H_0 + V_\varepsilon^m$ ,  $m = 0, \dots, n$ , where  $V_\varepsilon^m = \sum_{j=1}^m A_j^* K_\varepsilon S_j A_j$ . Since all but the last perturbations are either positive or infinitesimal (in which case one has to use Lemma 7.6), we can assume that  $A_m^* K_\varepsilon S_m A_m$  is relatively form bounded with uniform bound less than one with respect to  $H_\varepsilon^l$  with  $l < m$ .

Now, the result follows by applying the previous lemma with  $H = H^{m-1}$ ,  $\tilde{H} = H^m$ ,  $A = A_m$ ,  $S = S_m$  and  $B = B_l$ ,  $l = m + 1, \dots, n$  and letting  $m$  going up from 1 to  $n$ .  $\square$

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